INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Second Year, Second Semester, 2005-06
Statistics - II, Semesteral Examination, May 2, 2006

(8) 1. n independent trials, each with probability p of success, are carried out. Trials are then continued further (independently) until an additional success is obtained, this requiring S additional trials. Let the results be denoted as
$X_1, \ldots, X_n, X_{n+1}, \ldots, X_{n+S-1}, X_{n+S}$, where X_i is either 1 (success) or 0 (failure).
(a) What is $P(\sum_{i=1}^{n+S} X_i = k)$ for any integer k?
(b) What is minimal sufficient for p? Justify.
(c) Find the maximum likelihood estimate of p.

(5) 2. Let $X_1, \ldots, X_n$ be i.i.d. Uniform $(0, \theta)$, $\theta > 0$.
(a) What is the minimal sufficient statistic for θ? Is it complete?
(b) Show that $X_{(n)}$, the largest order statistic, is independent of X_2/X_1.

(4) 3. Suppose X can have density f_0 or f_1. To test at level α

$$H_0 : X \text{ has density } f_0 \qquad \text{versus} \qquad H_1 : X \text{ has density } f_1,$$

consider the test given by the following test function ϕ which satisfies $E(\phi(X)) = \alpha$ under H_0 and

$$\phi(x) = \begin{cases} 1 & \text{if } f_1(x)/f_0(x) > k; \\ 0 & \text{if } f_1(x)/f_0(x) < k, \end{cases}$$

for some $k > 0$. Show that this test is a most powerful test.

(5) 4. The density of the three-parameter Weibull distribution is

$$f(x; a, b, c) = \begin{cases} (a/b)((x-c)/b)^{a-1} \exp\left(-((x-c)/b)^a\right), & \text{if } x > c, \\ 0 & \text{otherwise,} \end{cases}$$

where $-\infty < c < \infty$, $a > 0$, $b > 0$. Let $X_1, X_2, \ldots, X_n$ be a random sample from this distribution. Suppose that a and c are known. Consider testing

$$H_0 : b \leq b_0 \text{ versus } H_1 : b > b_0.$$

Does a UMP level α test exist? Find it if it does, and indicate what statistical tables are to be used to determine the critical values of a level α test.

(8) 5. Let $X_1, X_2, \ldots, X_n$ be independent observations from $N(\mu, \sigma^2)$ where both μ and σ^2 are unknown. Consider testing $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 \neq \sigma_0^2$. Find the generalized likelihood ratio test for this at the significance level α.

(8) 6. The weekly number of fires, X, in a town has the $Poisson(\theta)$ distribution. The numbers of fires observed for five weekly periods were 0, 1, 1, 0, 0. Assume that the prior distribution on θ is
$\pi(\theta) \propto \theta^{-2} I_{(0.01,\infty)}$.
(a) Find a 90% HPD credible set for θ.
(b) Suppose that it is desired to test $H_0 : \theta \leq .15$ versus $H_1 : \theta > .15$. Find the posterior probability of H_0. What do you conclude from this?

(12) 7. Let $X_1, X_2, \ldots, X_n$ be i.i.d. $N(\mu.\sigma^2)$, where $\mu \geq 0$.
(a) Find the m.l.e. $(\hat{\mu}, \hat{\sigma}^2)$ of (μ, σ^2).
(b) Find the joint probability distribution of $\hat{\mu}$ and $\hat{\sigma}^2$.
(c) Find the asymptotic distribution of $\sqrt{n}(\hat{\sigma}^2 - \sigma^2)$ as $n \longrightarrow \infty$.
(d) Is $\hat{\sigma}^2$ asymptotically efficient? Justify.